Market Integration and Global Crashes *

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Abstract

We develop an equilibrium model of real and financial market integration in which real firms and financial actors independently decide on their investment into different locations (countries). We show that, in the presence of financial frictions, firms' real investment choices may become strategic complements, leading to multiple, self-fulfilling equilibria, as well as to real fragility, whereby a small change in one country's fundamentals triggers a large large change in real investment everywhere. This fragility may lead to a global crash in which severe underinvestment into countries with underdeveloped financial markets spills over all other countries. We show that such global crashes are particularly severe when frictions are sufficiently symmetric across countries. By contrast, with enough asymmetry, the economy is likely to end up in a local crash equilibrium in which countries with low real investment barriers suffer the most.

Keywords: Market Integration, Foreign Direct Investment, Frictions, Fragility, Crashes, Crises

JEL Classification Numbers: G01, G12, F62, F65, F36

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1 Introduction

The effects of international portfolio flows, foreign direct investment, and trade on economic growth and financial stability have become a major concern of policy makers all over the world. While increasing international integration, often referred to as globalization, is typically viewed as a positive trend, many economists have expressed concerns that increased integration has made the world economy more fragile, as economic shocks to one country may have a large impact on the world economy as a whole.[1] Such international spillovers may occur both through the financial markets channel and through the real investment channel, and it is often difficult to disentangle the two. The goal of this paper is to develop a tractable theoretical model for studying the interaction of real and financial integration. Our key contribution is to show how real economic linkages, such as foreign direct investment (FDI) or trade, contribute to propagating global economic downturns that originate in a local financial market.

To this end, we develop a parsimonious equilibrium model with multiple locations (countries) in which real firms and financial actors (shareholders) independently decide how much to invest into each country. We assume that there are barriers to real and financial integration: both real firms and financial investors face (location and destination-specific) frictions (costs) to investing abroad. We model the financial frictions as iceberg costs and assume that these costs are proportional to investors' foreign portfolio holdings. Importantly, this assumption implies that the cost of holding one share in foreign equity is fixed in the sense that it is independent of real investment choices of foreign firms. It is this structure of financial frictions that opens a channel for multiple self-fulfilling equilibria and leads to equilibrium fragility of international financial markets: in a "bad" equilibrium, real firms believe that their country's market will not have a scale necessary to justify the fixed costs

[1] See, for example, Caruana (2015) and
http://www.ft.com/cms/s/2/8514c0dc-17af-11e2-9530-00144feabdc0.html#axzz4CFrtj8bd

and attract large capital inflows, which depresses asset prices and real investment. A "good" equilibrium is characterized by the opposite pattern.

One of the key insights from our model is that real integration serves as a channel for international transmission of financial shocks and makes the global economy fragile: in a bad equilibrium, small shocks to countries with under-developed financial markets spill back into other countries and may cause a global crash.[2] The exact mechanism underlying the global fragility is subtle and is driven by substitution effects between real and financial investments. To illustrate this mechanism, consider an economy with two countries, the U.S. and China. Suppose that the only difference between the two countries is that financial investors face fixed costs of investing into the Chinese stock market. These costs preclude investors from attaining the efficient level of exposure to Chinese total factor productivity (TFP) shocks.[3] However, in the presence of real integration, U.S. firms invest directly into China and hence holding U.S. equity serves as a substitute for holding the Chinese stock market. This substitution effect increases the demand for U.S. equity, and drives U.S. stock prices and real investment up. We call it a positive spill-back effect. Absent barriers to real investment (such as iceberg costs for international trade), this effect is robust and is not sensitive to the fundamentals of the Chinese economy: Effectively, U.S. firms investment into China serves as a good substitute for a direct exposure to the Chinese stock market. However, in the presence of barriers to real investment, U.S. firms' investment into China is insufficient, and the substitution effect is not strong enough. As a result, the compensation that U.S. investors offer to U.S. firms for their real investment into China drops inversely proportionally to Chinese local real investment. In a bad equilibrium, Chinese local real investment is very

[2]There are two types of crashes that may occur in our model. The first one arises due to equilibrium fragility whereby a small change in the fundamentals of one country leads to a large change in the global economy. See, e.g., Allen and Gale (2004). The second one is similar to that in Obstfeld (1986), whereby a crash occurs when the economy switches from a good to a bad equilibrium.

[3]In addition to total factor productivity itself, these shocks could also encompass demand shocks, exposure to foreign currency risk, as well as other types of risks such as, e.g., political uncertainty.

low, and hence the global economy is fragile because the U.S. real investment is highly sensitive to Chinese fundamentals.[4]

The tractability of our model allows us to study a rich set of economic structures with multiple (more than two) countries and asymmetric real and financial frictions. To the best of our knowledge, our paper is the first to highlight the importance of asymmetries between countries. We show that the case of identical countries that is often studied in the literature exhibits a behavior that is non-generic and therefore may be misguiding. Namely, when countries are symmetric, there exists a continuum of equilibria with essentially arbitrary properties. However, even arbitrarily small degrees of asymmetry serve as an equilibrium selection mechanism: for generic parameter values, there are at most two equilibria, a good one and a bad one. Furthermore, asymmetries in real frictions preclude the emergence of "very bad" global crashes.

2 Literature Review

A dramatic increase in international trade flows over the past decades (the so-called "globalization") has attracted a lot of attention in academic research. For example, Rey (2013) provides evidence of a "global financial cycle" by showing that a tightening liquidity constraint - as proxied by VIX - typically leads to a drop in most types of cross-border capital flows. Interestingly enough, she finds that FDI flows is a notable exception as it often co-moves positively with VIX. More generally, portfolio flows and FDI are negatively related in the data, suggesting that the two are (at least to some extent) substitutes. These observations are consistent with the predictions of our model: When financial constraints tighten, real firms may try to substitute and create direct exposure to foreign TFP shocks though FDI. See, Errunza and Senbet (1981). Indeed, it is commonly viewed that globalization exposes

[4] Anecdotal evidence suggests the U.S. economy is indeed excessively sensitive to Chinese economic news. For example, in January 2016 the Dow Jones Industrial Average fell nearly 4% in response to a fall in the Chinese stock market. See http://www.wsj.com/articles/global-stocks-fall-on-china-volatility-1452156855

domestic economies to foreign productivity shocks (see, for example, Goldberg and Pavcnik, 2007; Melitz and Redding, 2014), in agreement with the way we define real integration in our model. Interestingly enough, Petzev, Schrimpf, and Wagner (2016) find that, despite a gradual increase in international correlations, local risk factors still play an important role in pricing local stocks. They interpret their findings as an evidence for increased real integration co-existing with stock markets fragmentation, as in our model.

There is significant empirical evidence suggesting that financial market frictions may have large real effects. For example, van Binsbergen and Opp (2016) find that financial market imperfections can cause large and persistent deviations in real investments and that the real costs of incomplete risk sharing and other financial market imperfections are quite large; Caldara, Fuentes-Albero, Gilchrist, and Zakrajsek (2016) provide empirical evidence that the Great Recession was to a large extent driven by an interaction of financial and uncertainty shocks. Dedola and Lombardo (2010), Devereux and Yetman (2010), Devereux and Sutherland (2011), Kollmann, Enders, and Muller (2011), Kollmann (2013), and Lagarde (2016) highlight the role that financial markets play in increasing international co-movement of real variables. Imbs (2010) provides strong empirical evidence that real and financial integration mutually re-inforce each other. His findings suggest that for developed economies, financial integration leads to real integration, while for developing economies causality seems to go in the opposite direction. Our model's predictions agree with most of these empirical findings: In fact, we show that even very small amounts of financial frictions can seriously hamper real investment and lead to significant co-movement and global crashes.

Our key results are driven by the assumption of barriers (costs) to international diversification. While it is possible to micro-found them using fixed monitoring costs borne by financial intermediaries, one could also view them either as some form of transaction tax[5]

[5]This could be an explicit tax, as in Martin and Rey (2006), or an implicit tax due to asymmetric information, as in Garleanu, Panageas, and Yu (2016).

or as a reduced form model for home bias.[6] Several papers, starting with Stulz (1981) with Errunza and Losq (1985, 1989), investigate the effects of barriers to international financial integration on equilibrium asset prices.[7] As in our model, Garleanu, Panageas, and Yu (2015) also consider a multi-country economy with costly financial integration and no real investment. In their model market access costs are assumed to fixed and only depend on the number of countries in which an agent invests. Garleanu, Panageas, and Yu (2015) show that, due to a complementarity between participation and leverage decisions, the market equilibrium may exhibit diverse leverage and participation choices across investors, even though investors are ex-ante identical. They show in their model that prices are subject to global crashes and may exhibit large swings in response to small local shocks. Our model differs from Garleanu, Panageas, and Yu (2015) in several aspects. First, in our model market access costs are not fixed, but are proportional to international portflio holdings. This cost structure makes the model more tractable and allows us to handle arbitrary geographical structures and arbitrary asymmetric market access costs. While our model also generates international shock spillovers and global crashes, the mechanism is very different and operates through investment complementarities across real firms.

Our model features multiple equilibria that arise through the mechanism of self-fulfilling beliefs. This relates our paper to the large literature on self-fulfilling crises. For example, Bacchetta, Tille, and van Wincoop (2012) and Bacchetta and van Wincoop (2013) show how panics can arise due to self-fulfilling shifts in beliefs about risk; Bacchetta and van Wincoop (2015) show that self-fulfilling global business cycle panics may arise in a two-country model in the presence of price rigidity, financial markets autarky, and sunspot

[6]Empirical evidence for home bias in investors' portfolios is vast. See, for example, Coval and Moskowitz (1999); Ivkovic and Weisbenner (2005); Baik et al. (2010); Couerdacier and Rey (2012); Bernile et al. (2015).

[7]Naturally, integration influences the (endogenous) structure of financial markets and financial intermediation. For example, Martin and Rey (2000) show how integration is linked to an endogenous breadth of financial markets. There is also a growing literature that emphasizes the role of frictions in foreign exchange markets in shaping international capital flows (both real and financial). See, for example, Gabaix and Maggiori (2015).

shocks that generate global panics. Perri and Quadrini (2013) develop a dynamic, two-country model with credit market frictions to show that a global liquidity shortage induced by pessimistic self-fulfilling expectations can generate crises. They also show that stronger international financial integration leads to crises that are less frequent but, when they hit, they are larger and more synchronized across countries. In addition, there are many papers that show that the financial sector plays a role in the propagation of non-financial shocks: For example, this may occur due to time-varying uncertainty and liquidity frictions (costly access to finance for real firms) as in Arellano, Bai and Kehoe (2012), or through interbank crises as in Boissay, Collard, and Smets (2012) and Cetorelli and Goldberg (2012). While our also features an impact of financial integration on global crises, the channel through which this impact operates is very different from the channels studied in the above-mentioned papers. Namely, in our model global crises occur due to an interaction between cross-border real investment by firms and cross-border financial investment by market participants. To the best of our knowledge, the fact that the exposure of real firms' profits to foreign TFP shocks depends on the degree of financial integration and may lead to global crashes is new to our model.[8]

Similarly to our paper, Martin and Rey (2006) also study the joint effects of real and financial integration on equilibrium real investment. They consider a two-country model in which international trade and diversification are subject to frictions (transaction costs), with one of the countries (emerging market) being more productive that the other. They show that financial frictions may lead a self-fulfilling crash in the emerging market associated with a collapse in the demand for goods and assets. Furthermore, they show that, while trade globalization always has a positive effect on the emerging market, financial globalization may not, especially when trade costs are high. In stark contrast to our model, in their model crashes can only occur in the emerging market, and global crashes never happen.

[8]The literature on global games (see Morris and Shin, 1998) highlights how even small amounts of information asymmetry may eliminate multiple equilibria and lead to equilibrium uniqueness. Investigating the role of asymmetric information for market integration is an important topic for future research.

In a recent study, Barro, Loualiche, and Sauvagnat (2015) investigate how real integration and globalization (measured using the international shipping costs of U.S. firms) is reflected in asset prices. They find empirically that firms in lower shipping costs industries carry a significant risk premium, and then develop a theoretical model that explains this risk premium through the displacement risk due to import competition. In addition, they find that foreign productivity shocks are perceived as bad news for the marginal U.S. investor. The theoretical results in Barro, Loualiche, and Sauvagnat (2015) are based on the assumption of no international risk sharing, which is an extreme form of market fragmentation. In our model, we show that even arbitrarily small amounts of fragmentation may have very large real effects on the economy. Investigating the interactions between the product market competition channel from Barro, Loualiche, and Sauvagnat (2015) with the degrees of real and financial integration is an important direction for future research.

Another paper related to our is Parlour and Walden (2011), who consider a single country economy populated by CARA investors and real firms; their model also features an interaction between investors' portfolio choices and real investment in the presence of frictions. In their model, frictions arise due to workers' moral hazard and endogenous effort choice.

3 Model

We consider a two-date economy populated by real firms and financial investors. At time $t = 0$, investors trade in financial markets rationally anticipating firms' investment policies. Simultaneously, firms make their investments, maximizing firm value and taking market prices as given.

We assume that the economy consists of N locations $i = 1, \cdots, N$ (also called countries in the sequel). Each location comprises a continuum of identical firms and a continuum of identical investors. A firm in location i decides on the scale $q_{i\to j}$ of capital investment into location j for each $j = 1, \cdots, N$. We assume that the investment cost is convex.[9] The supply of shares in each firm is normalized to $1/N$. We denote by $S = (S_i)_{i=1}^N$, $S_i = 1/N$ for all i, the vector of total supply. Investing at vector of scales $q_i = (q_{i\to j})_{j=1}^N > 0$ costs $I_i(q_i) \equiv \sum_{j=1}^N (a_{i,j}^R q_{i\to j} + 0.5 b_{i,j} q_{i\to j}^2)$ for some location-specific parameters $a_{i,j}^R$, $b_{i,j} > 0$, $i, j = 1, \cdots, N$. Given the choice of investment scales q_i, the firm generates a random revenue (dividend) of

$$D_i \; = \; \sum_{j=1}^N q_{i\to j} \, X_j \tag{1}$$

at time $t = 1$. Here, the random variable X_j stands for total factor productivity in location j in the case when the firm in location i actually produces in location j (say, through a foreign direct investment) and/or demand shocks in location j for the goods produced by the firm i in the case when the firm is only selling (and not producing) its goods in location j. It may also capture exposure to foreign currency risk, as well as other types of risks such as, e.g., political uncertainty. We assume that the vector $X = (X_i)_{i=1}^N$ is normally distributed, $X \sim N(\bar{X}, \Sigma^X)$, and we denote by $Q = (q_{i\to j})_{i,j=1}^N$ the matrix of investment scales.

Each financial investor in location i chooses a portfolio π_i to maximize a CARA expected utility

$$E[-e^{-\gamma_i \, c}]$$

subject to a short-sales constraint $\pi_i \geq 0$. We will only consider equilibria in which short-sale constraints do not bind. Note that we allow for potential differences in risk aversions across locations. One should not necessarily interpret this as differences in preferences: Rather, $1/\gamma_i$ should be viewed as a measure of the total risk bearing capacity of country i. For example,

[9]That is, consistent with the empirical evidence, marginal investment costs are increasing in investment level.

if a country i has a larger financial system, local investors should have a larger risk-bearing capacity.

We assume that there exist financial frictions making financial investments costly. Namely, we assume that an investor in location i holding a position $\pi_{i,j}$ in location j incurs a cost of $a^F_{i,j}\pi$.[10] We denote $a^F_i = (a^F_{i,j})_{j=1}^N \in \mathbb{R}^N$ and $A^F = (a^F_{i,j})_{i,j=1}^N \in \mathbb{R}^{N\times N}$. In addition, we assume that each investor type always invests equally into all firms in a given location: Thus, $\pi_{i,j}$ is the holding of the stock market index of country i by investors in country j.[11]

4 Equilibrium

Given the CARA-normal setup and assuming that the short-sale constraints do not bind, investors' portfolio solves the standard mean-variance optimization problem and is given by

$$\pi_i \;=\; \gamma_i^{-1}\Sigma_D(Q)^{-1}\left(\bar{D}(Q) \;-\; e^r P \;-\; a_i^F\right)\,,$$

where $\Sigma_D(Q)$ is the covariance matrix of firms' dividends, and $\bar{D}(Q)$ is the vector of expected dividends. By (1), we have

$$\Sigma_D(Q) \;=\; Q\Sigma^X Q^T\,,\;\; \bar{D}(Q) \;=\; -I(Q) + Q\bar{X}\,.$$

Writing down market clearing, we arrive at the following result.

[10]This friction can be micro-founded by assuming that investors need to monitor the real firms they are invested in. Namely, suppose that each investor always holds one share in each location-j firm he is invested in. Since all firms in each location are identical, holding one share in each of the π firms is equivalent to holding π shares in one firm. Assuming that the investor needs to pay a cost of $a_{i,j}$ to monitor each firm justifies the assumed cost structure.

[11]This assumption is standard in the literature. See, for example, Martin and Rey (2006) and Parlour and Walden (2011). It implies that a given firm cannot impact its own price of risk by deviating and attracting investors through offering a better risk-return profile than its competitors. Effectively, it means that each particular firm is "small" and behaves non-strategically: Indeed, if a local firm could credibly commit to deviating for other local firms' policies, it would behave strategically and would be able to influence stock prices of all other firms.

Proposition 1 *Suppose that the matrix Q is invertible. Then, equilibrium asset prices are given by*

$$P \;=\; -e^{-r}I(Q) \;+\; QM$$

where $M = (M_i)_{i=1}^N$ are the prices of local shocks X_i, given by

$$M \;=\; Q^{-1}e^{-r}\,\left(Q\bar{X} - \bar{a}^F - \bar{\gamma}\,\Sigma_D(Q)S\right)$$

with $\bar{\gamma} = \left(\sum_i \gamma_i^{-1}\right)^{-1}$ and $\bar{a}^F = \bar{\gamma}\sum_i \gamma_i^{-1}a_i^F$. Equilibrium portfolio holdings are given by

$$\pi_i \;=\; \gamma_i^{-1}\bar{\gamma}\,S \;+\; \gamma_i^{-1}\Sigma_D(Q)^{-1}(\bar{a}^F - a_i^F)\,.$$

As Proposition 1 shows, in equilibrium financial frictions get aggregated and their total impact on stock prices is given by the risk aversion-weighted cost $\bar{a}^F$. In particular, it does not matter which investor class faces the cost a^F, what matters is the weighted average cost borne by investors. Furthermore, in equilibrium, investors holds the market portfolio plus a correction that accounts for the deviation of their marginal cost a_i^F from the market-average cost $\bar{a}^F$. In particular, not surprisingly, if all investors face the same costs of investing into a given location, all investors hold the market portfolio.

As is common in the literature, we assume that real firms use equilibrium cost of capital M inferred from the market clearing prices to make optimal investment decisions.[12] Note that invertibility of the matrix Q is crucial for equilibrium existence: If Q is singular, prices

[12]In equilibrium, barriers to international trade imply that different agents hold different portfolios: In a way, financial markets are imperfectly integrated, and therefore agents from different locations would disagree about the opportunity cost of real investment for a given firm. As a justification for using the prices M, we assume is that there exists a small mass of speculators (arbitrageurs) who can costlessly access international markets, and can thus do the arbitrage trade that ensures that M_i is indeed the price of the shock X_i. When the mass of such arbitrageurs is sufficiently small, their impact on equilibrium prices will be negligible, validating our equilibrium characterization.

of local shocks X_i cannot be uniquely recovered from stock prices, and equilibrium is not defined.

Given the prices $(M_i)_{i=1}^N$, the market value of investing at the scale q in location j is given by qM_j, and the maximization problem of a firm in location i is given by

$$\max_{q_i \in \mathbb{R}_+^N} \{-I_i(q_i) \;+\; \sum_i q_{i\to j} M_j\}\,.$$

The corresponding optimal real investment is given by

$$q_i \;=\; B_i^{-1} \max\{(M \;-\; a_i^R), 0\}\,.$$

Let $A^R \;=\; (a_{i,j}^R)_{i,j=1}^N$ be the matrix of real investment costs, and define the matrix $\bar{X}^a$ of effective expected real investment payoffs via

$$\bar{X}^a \;\equiv\; e^{-r}\bar{X} - (A^R)^T\,.$$

Here, we abuse the notation and use $\bar{X}$ to denote the matrix with identical columns given by $(\bar{X}_i)_{i=1}^N$; similarly, we use S to denote the $N \times N$ matrix with all elements equal to $1/N\;:\;\; S \;=\; N^{-1}\mathbf{1}_{N\times N}$. Finally, we use $\bar{A}^F$ to denote the matrix with identical columns given by $\bar{a}^F$, we define $\hat{B} \;=\; (b_{i,j})$, and we use $\circ$ to denote the element-wise (Kronecker) product of matrices. Then, we arrive at the following equilibrium characterization.

Theorem 2 *The set of equilibria coincides with the set of solutions Q to the system*[13]

$$\hat{B}^T \circ Q^T \;=\; \max\{\bar{X}^a - e^{-r}Q^{-1}\bar{A}^F - e^{-r}\bar{\gamma}\,\Sigma^X\,Q^T S, 0\}, \tag{2}$$

such that $\pi_i \geq 0$ for all i.

[13] Note that the vector q_i corresponds to the i'th column of the matrix Q^T. The max for matrices is understood element-by-element.

Equation (2) highlights the key role that is played by the financial frictions $\bar{A}^F$ in our model: while the implicit marginal costs due to risk, $e^{-r}\bar{\gamma}\,\Sigma^X\,Q^T S$, are monotonically increasing in investment scale Q, the marginal costs $Q^{-1}\bar{A}^F$ due to financial frictions are monotonically decreasing in Q. Indeed, if all real firms choose to invest a lot, their total exposure to risk increases, and financial investors require a higher compensation for this additional risk, driving equilibrium risk premia up and discouraging real investment. At the same time, the fixed financial costs $\bar{A}^F$ make large real investment scale attractive for financial investors because it reduces the marginal cost $Q^{-1}\bar{A}^F$ they effectively pay per unit of real investment. In the limit when the investment scale Q drops all the way to zero, risk aversion-driven marginal costs $e^{-r}\bar{\gamma}\,\Sigma^X\,Q^T S$ vanish, while the friction-driven marginal costs $Q^{-1}\bar{A}^F$ explode, implying that real investment choices become strategic complements across firms when Q is sufficiently small. As we show below, these strategic complementaries lead to the emergence of self-fulfilling low-risk, low-investment equilibria. Everywhere in the sequel, we restrict our attention to the economically meaningful equilibria in which all elements of the matrix Q are strictly positive. In this case, the max operator in (2) is redundant.

5 Spillovers, Spill-backs, and Fragility with Two Countries

In this section we consider three special cases that allow us to highlight some of the key mechanisms at work in our setting. We assume that there are $N = 2$ countries. Furthermore, we assume that the first country (referred to as U.S. in sequel) has well developed financial markets, so that the financial investment barriers are negligible (that is, $\bar{a}_1^F = 0$), while there exist non-negligible financial frictions (that is, $\bar{a}_2^F > 0$) for investors willing to invest into the second country (referred to as China in the sequel). For the ease of notation, we also set interest rate $r = 0$.

To zero in on the mechanism that leads to multiple equilibria, consider first the case when neither real nor financial actors can invest abroad. In this case, (2) implies that equilibrium real investment $q_i = q_{i,i}$ within each country satisfies

$$b_{i,i}\, q_i \;=\; \bar{X}^a_{i,i} - q_i^{-1} \bar{a}^F_i - 0.5\gamma_i\, \Sigma_{i,i}\, q_i\,,\;\; i = 1, 2\,,$$

and we arrive at the following result.

Proposition 3 *Suppose that the countries are fully segmented, and that* $\bar{X}^a_{i,i} > 0,\;\; i = 1, 2$, *so that investment has a positive NPV. Then, an equilibrium exists if and only if*

$$\bar{a}^F_2 \;\le\; \frac{(\bar{X}^a_{2,2})^2}{4(b_{2,2} + 0.5\gamma_2 \Sigma_{2,2})}$$

If this inequality is strict, there are exactly two equilibria with Chinese local investment q_2 *given by*

$$q_{2,\pm} \;=\; \frac{\bar{X}^a_{2,2}}{2(b_{2,2} + 0.5\gamma_2 \Sigma_{2,2})} \;\pm\; \alpha_2^{1/2}\,,$$

with

$$\alpha_2 = \frac{(\bar{X}^a_{2,2})^2 - 4\bar{a}^F_2 (b_{2,2} + 0.5\gamma_2 \Sigma_{2,2})}{4(b_{2,2} + 0.5\gamma_2 \Sigma_{2,2})^2}\,.$$

These equilibria exhibit opposite comparative statics: while the good equilibrium $, q_{2,+},$ *is monotone decreasing in* $a^R,\; a^F,\; b,\; \gamma_2,$ *and* $\Sigma_{2,2}$, *the bad (crash) equilibrium* $q_{2,-}$ *is monotone increasing in the same parameters. In the limit of vanishing financial frictions, the crash equilibrium* $q_{2,-}$ *features a severe underinvestment: Namely, we have* $q_{2,-} \to 0$ *as* $\bar{a}^F_2 \to 0$. *At the same time, the U.S. local real investment is given by* $q_1 = \frac{\bar{X}^a_{1,1}}{b_{1,1} + 0.5\gamma_1 \Sigma_{1,1}}$.

As Proposition 3 shows, even arbitrarily small amounts of financial frictions lead to the emergence of multiple equilibria, including a "crash" equilibrium featuring a severe underinvestment. This crash equilibrium is self-fulfilling and is driven by the strategic complementarities of real investment choices across Chinese firms: If all firms choose to severely underinvest, holding firms' equities cannot justify the fixed cost $\bar{a}_2^F$. As a result, financial investors require a very large compensation for holding shares, increasing the firms' cost of capital and making it optimal for any given firm to severely underinvest. Not surprisingly, absent any kind of integration (neither real nor financial), there are no spillover effects: A crash in China has no impact on equilibrium investment in the U.S.

The next natural question is whether financial integration per se is sufficient to generate *real* spillover effects without any real integration (formally, absence of real integration can be guaranteed if $b_{i,j} = \infty$ for $i \neq j$). In this case, equilibrium equations take the form

$$\begin{aligned} b_{1,1} q_{1\to 1} &= \bar{X}^a_{1,1} - 0.5 e^{-r} \bar{\gamma} \sum_{j=1}^{N} \Sigma^X_{1,j} q_{j,j} \\ b_{2,2} q_{2\to 2} &= \bar{X}^a_{2,2} - e^{-r} q^{-1}_{2\to 2} \bar{a}^F_2 - 0.5 e^{-r} \bar{\gamma} \sum_{j=1}^{N} \Sigma^X_{2,j} q_{j,j} \, . \end{aligned} \tag{3}$$

As we can see from (3), absent real integration, Chinese real investment has a spill-over effect onto the U.S. real investment only to the extent the TFP shocks in the two countries are correlated ($\Sigma_{1,2} \neq 0$). This effect operates through the standard risk premium channel: assuming that the correlation is strictly positive ($\Sigma^X_{1,2} > 0$), an increase in the Chinese investment increases the overall amount of risk and, as a result, the level of risk premia in the economy, thereby reducing the propensity of U.S. firms to invest. Conversely, crashes in China have a positive effect on U.S. real investment.

Finally, we show that, real cross border investments, and the interaction of real and financial frictions open a channel for the Chinese crashes to spill back onto the U.S. economy. To single out the desired effect, we make a simplifying assumption that Chinese firms cannot invest into the U.S.[14] In addition, to isolate the spillover channel based on financial frictions, we assume zero correlation between TFP shocks $\Sigma_{1,2} = 0$.

Absent real investment from China into U.S. ($q_{2\to1} = 0$), we have

$$Q \;=\; \begin{pmatrix} q_{1\to1} & q_{1\to2} \\ 0 & q_{2\to2} \end{pmatrix} \;,\; Q^{-1} \;=\; \begin{pmatrix} q_{1\to1}^{-1} & -\frac{q_{1\to2}}{q_{1\to1}q_{2\to2}} \\ 0 & q_{2\to2}^{-1} \end{pmatrix} \;,$$

and equilibrium equations (2) can be rewritten as

$$\begin{aligned} q_{1\to1} &= b_{1,1}^{-1}\left(\bar{X}_{1,1}^{a} + \frac{q_{1\to2}}{q_{1\to1}q_{2\to2}}\bar{a}_2^F - 0.5\bar{\gamma}\Sigma_{1,1}^X q_{1\to1}\right) \\ q_{1\to2} &= b_{1,2}^{-1}\left(\bar{X}_{1,2}^{a} - \frac{\bar{a}_2^F}{q_{2\to2}} - 0.5\bar{\gamma}\Sigma_{2,2}^X (q_{1\to2} + q_{2\to2})\right) \\ q_{2\to2} &= b_{2,2}^{-1}\left(\bar{X}_{2,2}^{a} - \frac{\bar{a}_2^F}{q_{2\to2}} - 0.5\bar{\gamma}\Sigma_{2,2}^X (q_{1\to2} + q_{2\to2})\right) \;. \end{aligned} \tag{4}$$

Since both U.S. and Chinese firms use the same market prices M_2 to evaluate real investment into China, the differences between U.S. and Chinese real investment into China arise only from the differences in the real costs $b_{1,2}, b_{2,2}, a_{1,2}, a_{2,2}$, and we always have

$$b_{1,2}q_{1\to2} + a_{1,2}^R = b_{2,2}q_{2\to2} + a_{2,2}^R \,. \tag{5}$$

Substituting (5) into the last equation in (4), we arrive at

$$q_{2\to2} = b_{2,2}^{-1}\left((\bar{X}_{2,2}^{a} + \bar{\gamma}\Sigma_{2,2}(a_{1,2}^R - a_{2,2}^R)b_{1,2}^{-1}) - \frac{\bar{a}_2^F}{q_{2\to2}} - 0.5\bar{\gamma}\Sigma_{2,2}(b_{2,2}b_{1,2}^{-1} + 1)q_{2\to2}\right) \;. \tag{6}$$

[14]Formally, we assume that $b_{2,1} = \infty$. This assumption is made purely for illustrative purposes and is relaxed later on. Note that the opposite case (when only Chinese firms can invest into U.S.) cannot lead to any interesting effects: In this case, U.S. firms' local real investment only depends on the U.S. local cost of capital, which only interacts with the Chinese investment through the diversification channel.

Finally, solving the first equation in (4), we get[15]

$$
\begin{aligned}
q_{1\to 1} &= \frac{\bar{X}^a_{1,1} + \left((\bar{X}^a_{1,1})^2 + 4(q_{1\to 2}/q_{2\to 2})\bar{a}^F(b_{1,1} + 0.5\bar{\gamma}\Sigma_{1,1})\right)^{1/2}}{2(b_{1,1} + 0.5\bar{\gamma}\Sigma_{1,1})} \\
&= \frac{\bar{X}^a_{1,1} + \left((\bar{X}^a_{1,1})^2 + 4\left(\frac{b_{2,2}}{b_{1,2}} - \frac{\bar{a}^R_{1,2} - a^R_{2,2}}{b_{1,2}q_{2\to 2}}\right)\bar{a}^F(b_{1,1} + 0.5\bar{\gamma}\Sigma_{1,1})\right)^{1/2}}{2(b_{1,1} + 0.5\bar{\gamma}\Sigma_{1,1})}.
\end{aligned} \tag{7}
$$

From (6) we see that U.S. firms' costs to real investment into China (that is, $a^R_{1,2} - a^R_{2,2} > 0$) enhance Chinese local real investment. More subtly, frictions that impede financial investment into China increase U.S. local real investment. This happens because investors require a lower overall cost of capital from the U.S. firms that allow them to get exposure to China's TFP shocks through their real investment to China. In fact, U.S. local investment is always higher in the presence of frictions: since $q_{1\to 2},\ q_{2\to 2} > 0$, we have from (7)

$$
q_{1\to 1} > q^*_{1\to 1} = \frac{\bar{X}^a_{1,1}}{b_{1,1} + 0.5\bar{\gamma}\Sigma_{1,1}}
$$

As in Proposition 3, equation (6) is a quadratic equation that has two solutions, and the smaller one corresponds to a crash equilibrium with very low China local real investment values. When a crash in China occurs, the architecture of real international investments unravels.

First, U.S. real investment into China $q_{1\to 2}$ falls in line with Chinese local investment $q_{2\to 2}$, as can be seen from (5). This is due to the soaring compensation required by investors to hold Chinese shares that spills over into the price of real U.S. investment into China through financially integrated markets.

Second, when $a^R_{1,2} - a^R_{2,2} > 0$, a crash in China also leads to a drop in U.S. local investment. As we explain above, U.S. firms are compensated for providing exposure to China's TFP shocks free of financial costs associated with investing in Chinese stocks. Absent

[15] The second solution to this quadratic equation is negative and hence cannot be an equilibrium. Note also that $q_{1\to 1} = 0$ cannot be an equilibrium either because then the matrix Q would be singular.

any impediments to international real investment, $a^R_{1,2} = a^R_{2,2}$, U.S. real firms continue to play this role after the crash and scale their investment into China proportionally to that of Chinese firms. In agreement with this intuition, formula (7) shows that in this case U.S. local investment is independent of the Chinese internal investment. Conversely, barriers to real investment into China (as measured by $\bar{a}^R_{1,2} - \bar{a}^R_{2,2} > 0$) make U.S. market exposure to Chinese TFP shocks an imperfect substitute for the exposure gained by holding shares in the Chinese stock market. In this case, the U.S. firms underinvest in China and do relatively more so during the crash. Through this channel, U.S. local investment suffers during the crash as the compensation U.S. firms get for providing exposure to China drops. Indeed, examining (4), we can see that U.S. investment $q_{1\to 1}$ depends on the investment into China through the quotient $q_{1\to 2}/q_{2\to 2}$, and is monotonically increasing in the latter. As $q_{1\to 2}/q_{2\to 2}$, falls with $q_{2\to 2}$, so does $q_{1\to 1}$.

Our results stand in stark contrast with the results of Martin and Rey (2006), who show that financial frictions may lead to crashes in emerging markets, but can never lead to global crashes.[16] That is, while crashes may occur in China in the model of Martin and Rey (2006), these crashes never spillover to more developed markets in their model. The major difference between our model and that of Martin and Rey (2006) comes from the definition of real integration. Namely, in their model, firms only invest locally, and real integration is defined through international trade that is performed directly by consumers. Thus, local TFP shocks have no impact on the risk profile of foreign firms. By contrast, in our model real integration is defined through the exposure of local firms to foreign TFP shocks: As Barro, Loualiche, and Sauvagnat (2015) show, this exposure is important and is priced in the cross-section of U.S. firms. A key insight from our model is that *real and financial integration are substitutes*: if it is costly for U.S. investors to acquire shares in the Chinese stock market, they compensate U.S. firms in terms of the required risk premia for investing into China and

[16] Martin and Rey (2006) define an emerging market as the country with a less efficient real production technology.

thereby creating an indirect exposure to Chinese TFP shocks. Real firms try to substitute and invest more into China, but this substitution may collapse when they face real barriers of entry to Chinese markets.

6 General Properties of Equilibria

As we explain above, we only consider equilibria with strictly positive investment. In this case, equilibrium equations (2) take the form

$$Q(\hat{B} \circ Q^T) \;=\; Q\bar{X}^a - e^{-r}\bar{A}^F - e^{-r}\bar{\gamma}\, Q\Sigma^X\, Q^T S\,. \tag{8}$$

Throughout this section, we make the following assumption.

Assumption 1 *Quadratic real costs are symmetric across countries. Namely, there exists a vector $c = (c_j)_{j=1}^N \in \mathbb{R}_+^N$ such that $b_{i,j} = c_j,\ i,j = 1,\cdots,N$. Furthermore, there are barriers to international real investment in the sense that $a_{i,j}^R > a_{j,j}^R$, or, equivalently $\bar{X}_{j,i}^a < \bar{X}_{j,j}^a$ for any $i \neq j$*

Under Assumption 1, equation (8) simplifies because in this case we always have

$$\hat{B} \circ Q^T = CQ^T.$$

Everywhere in the sequel, we will use

$$\bar{q}_{\to j} \equiv \frac{1}{N}\sum_{i=1}^N q_{i\to j},\ j = 1,\cdots,N$$

to denote the total equilibrium investment into a given country j. We also define

$$\Gamma \equiv \ (C + e^{-r}\bar{\gamma}\Sigma^X)^{-1/2}\bar{X}^a.$$

Finally, we will use * to denote equilibrium quantities in the case without financial frictions. Everywhere in the sequel we make the following assumption.

Assumption 2 *The matrix* $\bar{X}^a - e^{-r}\bar{\gamma}\Sigma^X(C+e^{-r}\bar{\gamma}\Sigma^X)^{-1}\,\bar{X}^a S$ *has strictly positive elements and is non-degenerate.*

The following is true.

Proposition 4 *Suppose that there are no financial frictions, so that* $\bar{A}^F = 0$*. Then, the matrix of equilibrium investments,* Q^**, is given by*

$$(Q^*)^T \;=\; C^{-1}(\bar{X}^a \;-\; e^{-r}\bar{\gamma}\Sigma^X(C+e^{-r}\bar{\gamma}\Sigma^X)^{-1}\,\bar{X}^a S)\,,$$

while the total inward investment satisfies

$$\bar{q}^*_{\to i} \;=\; \sum_j \Gamma_{i,j}\,,\;\; i = 1,\cdots,N\,.$$

The result of Proposition 4 is very intuitive: absent financial frictions, equilibrium investment is determined by the optimal level $C^{-1}\bar{X}^a$ net of the compensation for the aggregate risk. Markets are fully integrated, all investors agree on the valuations of all assets, and equilibrium investment level is efficient, in the sense that it coincides with that chosen by a social planner maximizing the total welfare in the economy.

Suppose now that there are $N = 2$ countries (U.S. and China), and that only country 2 (China) is subject to financial frictions, that is, $\bar{a}_1^F = 0 < \bar{a}_2^F$. Let also $y_1 \equiv b_{2,1}^{-1}(\bar{X}^a_{1,1} - \bar{X}^a_{2,1}) > 0$ and $y_2 \equiv b_{2,2}^{-1}(\bar{X}^a_{2,2} - \bar{X}^a_{1,2}) > 0$ be the normalized barriers to real investment. We also define the matrix $F = (C + e^{-r}\bar{\gamma}\Sigma^X)^{-1}$ and

$$\alpha \equiv \big((y_1 q^*_{1,2} - y_2 q^*_{1,1} + 2F_{1,2}\bar{a}_2^F)^2 + 4(y_2 q^*_{1,2} - F_{2,2}\bar{a}_2^F)(y_1 q^*_{1,1} + F_{1,1}\bar{a}_2^F)\big)^{1/2}\,.$$

The following proposition characterizes equilibria in the two country case.

Proposition 5 *Under Assumption 1, suppose that* $\bar{a}_1^F = 0$. *Then, there can be at most two equilibria,* $(q^+_{i,j})_{i,j=1}$ *and* $(q^-_{i,j})^2_{i,j=1}$, *given by*

$$q^{\pm}_{1,1} \;=\; q^*_{1,1} + \frac{F_{1,1} - F_{1,2}Z_{\pm}}{Z_{\pm}y_2 + y_1}\bar{a}_2^F,\; q^{\pm}_{1,2} = q^*_{1,2} + \frac{F_{1,2} - F_{2,2}Z_{\pm}}{Z_{\pm}y_2 + y_1}\bar{a}_2^F$$
$$q^{\pm}_{2,1} = q^{\pm}_{1,1} - y_1,\; q^{\pm}_{2,2} = q^{\pm}_{1,2} + y_2\,,$$

where

$$Z_{\pm} \;=\; \frac{-(y_1 q^*_{1,2} - y_2 q^*_{1,1} + 2F_{1,2}\bar{a}_2^F) \pm \alpha}{2(y_2 q^*_{1,2} - F_{2,2}\bar{a}_2^F)}$$

always satisfies $Z_{\pm} = \frac{q^{\pm}_{1,1}}{q^{\pm}_{1,2}}$. *Furthermore, given a* $K = +, -$, *the matrix* $Q^K = (q^K_{i,j})^2_{i,j=1}$ *is a true equilibrium if* $q^K_{i,j} > 0$ *for all* $i, j = 1, 2$.

To isolate the effects of frictions on international spillovers, assume for simplicity that the TFP shocks are uncorrelated across countries, so that $\Sigma^X_{1,2} = 0$, which is in turn equivalent to $F_{1,2} = 0$. In this case, as in the setting discussed in the previous section, the U.S. economy always profits from Chinese frictions, while China always suffers: $q_{1,1} > q^*_{1,1}$, $q_{1,2} < q^*_{1,2}$. Furthermore, equilibrium is fragile if both $Z_{\pm}y_2$ and y_1 are sufficiently small, and y_2 is non-zero: indeed, $\frac{\partial q_{1,1}}{\partial \alpha} = -\frac{y_2}{(y_2 Z_{\pm} + y_1)^2}\frac{\partial Z_{\pm}}{\partial \alpha}$ for any parameter α of the Chinese economy. The intuition is as above: absent barriers to real investment into China, U.S. firms are able to serve as an efficient substitute for U.S. investors' exposure to Chinese TFP shocks. At the same, quite surprisingly, for the equilibrium to be fragile we also need that real barriers to investment into U.S. (i.e., y_1) be sufficiently small. This is a *real entanglement effect*: When it is easy for Chinese real firms to invest into U.S., they become attractive for U.S. investors, but at the same time they make U.S. investors portfolios more sensitive to shocks to Chinese fundamentals. The following corollary characterizes equilibria for different parameter regions.

Corollary 6 *Under Assumption 1, suppose that* $\bar{a}_1^F = \Sigma_{1,2}^X = 0$. *Then,*

- *If* $y_2 q_{1,2}^* > F_{2,2}\bar{a}_2^F$ *then there always exists a unique equilibrium given by* $(q_{i,j}^+)$. *This equilibrium is fragile: (i) When* $\bar{a}_2^F \uparrow y_2 q_{1,2}^* F_{2,2}^{-1}$, *we have* $q_{1,1} \downarrow q_{1,1}^*,\ q_{1,2} \to 0$; *(ii) When* $\bar{a}_2^F \uparrow y_2 q_{1,2}^* F_{2,2}^{-1}$ *and* $y_1 q_{1,2}^* \to y_2 q_{1,1}^*$, *we have* $\frac{\partial q_{1,1}}{\partial \bar{a}_2^F} \to \infty$.

- *If* $y_2 q_{1,2}^* < F_{2,2}\bar{a}_2^F$ *then*

  - *if either* $y_1 q_{1,2}^* < y_2 q_{1,1}^*$ *or* $y_1 q_{1,2}^* > y_2 q_{1,1}^*$ *and* $(y_1 q_{1,2}^* - y_2 q_{1,1}^*)^2 + 4(y_2 q_{1,2}^* - F_{2,2}\bar{a}_2^F)(y_1 q_{1,1}^* + F_{1,1}\bar{a}_2^F) < 0$ *then equilibrium does not exist;*

  - *if* $y_1 q_{1,2}^* > y_2 q_{1,1}^*$ *and* $(y_1 q_{1,2}^* - y_2 q_{1,1}^*)^2 + 4(y_2 q_{1,2}^* - F_{2,2}\bar{a}_2^F)(y_1 q_{1,1}^* + F_{1,1}\bar{a}_2^F) > 0$ *then there exist exactly two equilibria, the good equilibrium* $(q_{i,j}^+)$ *and the bad equilibrium* $(q_{i,j}^-)$, *satisfying* $q_{i,j}^+ > q_{i,j}^-,\ i,j = 1,2$.

    In the limit when $\bar{a}_2^F \downarrow y_2 q_{1,2}^* F_{2,2}^{-1}$, *we have*

    * $Z_+ \to \hat{Z} \equiv \frac{y_1 q_{1,1}^* + F_{1,1}\bar{a}_2^F}{y_1 q_{1,2}^* - y_2 q_{1,1}^*}$ *and* $q_{1,1}^+ \to q_{1,1}^* + \frac{F_{1,1}}{\hat{Z} y_2 + y_1}\bar{a}_2^F,\ q_{1,2}^+ \to q_{1,1}^+/\hat{Z}$;
    * $q_{1,1}^- \to q_{1,1}^*,\ q_{1,2}^- \to 0$.

    The bad equilibrium is fragile when $\bar{a}_2^F \uparrow y_2 q_{1,2}^* F_{2,2}^{-1}$ *and* $y_1 q_{1,2}^* \to y_2 q_{1,1}^*$.

Corollary 6 shows that, for sufficiently low frictions level $\bar{a}_2^F$, equilibrium is unique and it is actually a bad equilibrium that becomes fragile when $\bar{a}_2^F$ reaches a threshold level and the levels of real investment frictions, as captured by $y_1 q_{1,2}^*, y_2 q_{1,1}^*$ respectively, are sufficiently homogeneous across countries: in this parameter region, a slight increase in the cost $\bar{a}_2^F$ may lead to a full collapse of U.S. investment into China and to a simultaneous drop in the internal U.S. investment $q_{1,1}$ to the level $q_{1,1}^*$ corresponding to fully fragmented markets. At the same time, if we increase the cost $\bar{a}_2^F$ a bit further, a new equilibrium emerges, and this equilibrium features strictly higher real investment levels. Thus, from the point of view of a social planner, slightly increasing $\bar{a}_2^F$ might be beneficial: the economy will either stay in the bad equilibrium, or jump into the good equilibrium.

We now proceed to analyzing the general case with $N > 2$ countries. Multiplying (8) from left and right with the matrix S, and defining $\Theta \equiv SQ$, we arrive at the following equation:[17]

$$\Theta C \Theta^T \;=\; \Theta \bar{X}^a S - e^{-r} S \bar{A}^F S - e^{-r} \bar{\gamma}\, \Theta \Sigma^X\, \Theta^T\,, \tag{9}$$

where $C = \mathrm{diag}(c)$. The matrix Θ has a very intuitive interpretation: its elements represent average investment into a given country: $\Theta_{i,j} = \bar{q}_{\to j},\ i, j = 1, \cdots, N$. We also let

$$a^F_{\mathrm{tot}} \equiv e^{-r} \sum_i \bar{a}_i^F$$

be the total financial friction aggregated across all countries. The following theorem characterizes all solutions Θ to (9).

Theorem 7 *Under Assumption 1, let $C = \mathrm{diag}(c)$ define*

$$\alpha \;=\; -a^F_{\mathrm{tot}} + \frac{1}{4N} \sum_i (\bar{q}^*_{\to i})^2\,.$$

Equation (9) has real solutions if and only if $\alpha \geq 0$. Furthermore, for any equilibrium total inward investments vector $(\bar{q}_{\to i})_{i=1}^N$, there exists a vector $\beta = (\beta_i)_{i=1}^N \in \mathbb{R}^N$ satisfying $\sum_i \beta_i^2 = N^{-1}$ such that

$$(C + e^{-r}\bar{\gamma}\Sigma^X)^{1/2} (\bar{q}_{\to i})_{i=1}^N \;=\; \alpha^{1/2}\beta + 0.5 (q^*_{\to i})_{i=1}^N\,.$$

It is instructive to compare the full real integration case of Theorem 7 with the full

[17] Recall that we normalize the supply of each asset to $1/N$. Under this assumption, S is a projection: $S^2 = S$.

real segmentation case of Proposition 3. As Theorem 7 shows, financial frictions influence aggregate inward real investment through the quantity α. The structure of this quantity is very similar to that of the constant α_2 in Proposition 3: α measures the difference between the frictionless investment $\bar{q}^*_{\to}$ and the *total financial cost* a^F_{tot}, aggregated across all countries. The fact that only the total cost matters is important: It means that, due to real integration, all countries suffer in a global crash equilibrium even if only one of them is subject to financial frictions.

In the totally segmented case of Proposition 3, each country's economy behaves independently from the rest of the world, which corresponds to Theorem 7 when $N = 1$. In this case, $\beta^2 = 1$ implies that $\beta = \pm 1$ with $\beta = 1$ corresponding to a boom and $\beta = -1$ corresponding to a crash. Similar effects happen in the multi-country world with real integration: For example, if $\beta = \pm S = \pm \frac{1}{N}\mathbf{1}$, then all countries are equally influenced by equilibrium effects and we get two extreme equilibria

$$(\bar{q}_{\to i,\pm})_{i=1}^N \;=\; (C + e^{-r}\bar{\gamma}\Sigma^X)^{-1/2}\left(\pm\alpha^{1/2}S + 0.5(q^*_{\to i})_{i=1}^N\right). \tag{10}$$

At the same time, there may also exist equilibria in which different countries are asymmetrically influenced by aggregate financial frictions (as captured by a^F_{tot}). Since $\sum_i \beta_i^2 = 1/N$, we always have $|\beta_i| \le N^{-1/2}$. Furthermore, the only way this inequality can turn into an equality is when $\beta_j = 0$ for all $j \neq i$. This corresponds to an extremely asymmetric equilibrium in which only for one country in the world its total inward investment is influenced by the financial friction α^F_{tot}. We summarize this observations in the following corollary.

Corollary 8 *Normalized equilibrium investment in country i always satisfies*

$$-\alpha^{1/2}N^{-1/2} \;\le\; \{(C + e^{-r}\bar{\gamma}\Sigma^X)^{1/2}(\bar{q}_{\to j})_{j=1}^N\}_i - 0.5\bar{q}^*_{\to i} \;\le\; \alpha^{1/2}N^{-1/2},$$

and the inequality turns into an equality if and only if no other country is influenced by the financial frictions.

As we show in Proposition 3, in the absence of real integration, crash equilibria always co-exist with boom equilibria. A natural question to ask is whether these results still hold in the framework of Theorem 7. Since we only consider equilibria with strictly positive Q, this question reduces to understanding whether the bad equilibrium in (10) has positive aggregate investment SQ. While the freedom in choosing the vector β opens up a possibility for equilibria with different degrees of crashes, among these equilibria there is one "extreme global crash" equilibrium in which investment into all countries equally suffers from the crash. This equilibrium corresponds to $\beta = -\frac{1}{N}\mathbf{1}$. The following proposition introduces a necessary condition for the existence of such a global crash equilibrium.

Proposition 9 *[Symmetric countries are more prone to extreme global crashes] The following inequality is a necessary condition for the existence of the extreme global crash equilibrium:*

$$4a^F_{\text{tot}} \;>\; \frac{1}{N}\sum_i (\bar{q}^*_{\to i})^2 - \min_i (\bar{q}^*_{\to i})^2\,.$$

That is, frictionless real inward investment $\bar{q}^*_{\to i}$ must be sufficiently homogeneous across countries.

The intuition behind this result is as follows: in an extreme global crash equilibrium, all countries suffer equally from underinvestment. Due to real integration, the degree of this underinvestment is determined by the aggregate attractiveness of real investment, as measured by $\frac{1}{N}\sum_i (\bar{q}^*_{\to i})^2$. If, for a given country i, its total inward real investment $\bar{q}^*_{\to i}$ in the absence of frictions is too low, a global crash drives inward investment $\bar{q}_{\to i}$ to zero, which is impossible in equilibrium.[18]

[18]Indeed, if the total inward investment $\bar{q}_{\to i}$ is zero for some country i, investors cannot get any exposure

Theorem 7 characterizes total equilibrium inward investment $\bar{q}_{\to i}$ for a given country i. However, it is silent about the structure of outward investment across countries. To investigate the structure of this investment, we multiply equilibrium equation (8) by $(\mathrm{Id}-S)$ from both left and right. Defining $R \equiv (\mathrm{Id}-S)Q$, we get

$$RCR^T = R\,\bar{X}^a(\mathrm{Id}-S)\,. \tag{11}$$

Solutions to equation (11) are characterized explictly in the Appendix (see Lemma 12). Given such a solution and a solution Θ to (9), we obtain a candidate equilibrium $Q = SQ + (I-S)Q = \Theta + R$. This decomposition is intuitive: It represents the investment matrix $Q = (q_{i\to j})$ as the sum of the average investment per destination plus the deviations from this average. A candidate equilibrium is a true equilibrium if Q has non-negative elements non-negative and satisfies (8). Substituting $Q = \Theta + R$ into (8) and subtracting (11) and (9), we arrive at the following result.

Lemma 10 *A candidate equilibrium $Q = \Theta + R$ with nonnegative investments is a true equilibrium if and only if the following consistency conditions are fulfilled:*

$$\begin{aligned} \Theta C R^T &= \Theta \bar{X}^a(\mathrm{Id}-S) \\ R C \Theta^T &= R\bar{X}^a S \;-\; (\mathrm{Id}-S)\bar{A}^F \;-\; R\bar{\gamma}\Sigma^X\Theta^T\,. \end{aligned} \tag{12}$$

In general, verifying (12) is a non-trivial task. Here, we confine our attention to the class of equilibria in which $q_{i\to j}$ is independent of i. In the sequel we refer to such equilibria as symmetric. The following is true.

Proposition 11 *A symmetric equilibrium exists if and only if financial frictions are sym-*

to the TFP shock X_i and hence cannot invert the corresponding price of risk from stock prices. As a result, firms cannot evaluate the profitability of real investment into country i, and equilibrium is not well defined.

metric across countries (i.e., $\bar{a}_i^F$ is independent of i) and the set

$$\{\beta \perp \operatorname{span}(\bar{X}^a(\mathrm{Id}-S)) : \ (C+e^{-r}\bar{\gamma}\Sigma^X)^{-1/2}(\alpha^{1/2}\beta+0.5\bar{\Gamma}) \geq 0\} \tag{13}$$

is non-empty. Furthermore,

- *If real costs are symmetric (i.e., $a_{i,j}^R$ is independent of i) then there exists either none or a continuum of symmetric equilibria.*
- *If real costs are asymmetric (i.e., $a_{i,j}^R$ depends of i) then, for generic parameter values, there exist at most two symmetric equilibria.*

Proposition 11 has several important implications for the effect of asymmetries and financial frictions on the real economy. Intuitively, one would expect that it is the asymmetry in real investment costs that should be crucial for asymmetric real investment across countries. Quite surprisingly, Proposition 11 shows that this is not the case, and it is the asymmetry in financial frictions that drives the structure of outward investment across countries. Furthermore, while the standard, symmetric case commonly studied in the literature features very large non-uniqueness of equilibria, even a small degree of asymmetry in real costs serves as an equilibrium selection mechanism, reducing the number of equilibria to two. It is also interesting to understand how the firms in asymmetric countries cope with these asymmetries in a symmetric equilibrium: As formula (13) shows, they do it by selecting investment levels orthogonal to their real cost differentials. This orthogonality ensures that firms in different locations agree on the efficient investment levels across destinations.

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A Proofs

Proof of Theorem 7. Denote $Z \equiv SQ(C + e^{-r}\bar{\gamma}\Sigma^X)^{1/2}$ and $Y \equiv (C + e^{-r}\bar{\gamma}\Sigma^X)^{-1/2}\bar{X}^a S$. Then, we can rewrite (9) as

$$ZZ^T \;=\; ZY - e^{-r} S\bar{A}^F S\,. \tag{14}$$

Since matrices ZZ^T and $S\bar{A}^F S$ are both symmetric, we get that $ZY = Y^T Z^T$. Let $\tilde{Z} = Z - 0.5Y^T$. Then, we also have $\tilde{Z}Y = Y^T\tilde{Z}^T$. Substituting into (14), we get

$$\tilde{Z}\tilde{Z}^T \;=\; \frac{1}{4}Y^T Y - e^{-r}S\bar{A}^F S \;=\; 0.25S(\bar{X}^a)^T(C + e^{-r}\bar{\gamma}\Sigma^X)^{-1}\bar{X}^a S - e^{-r}S\bar{A}^F S$$

This means that $\tilde{Z} = \alpha^{1/2}S\mathcal{O}$ where $\mathcal{O}$ is an arbitrary orthogonal matrix and

$$\alpha \;=\; \frac{1}{N}\sum_{i,j}\left\{0.25(\bar{X}^a)^T(C + e^{-r}\bar{\gamma}\Sigma^X)^{-1}\bar{X}^a - e^{-r}\bar{A}^F\right\}_{i,j}\,.$$

Now, given $\tilde{Z}$, we recover

$$SQ = Z(C+e^{-r}\bar{\gamma}\Sigma^X)^{-1/2} = S(\alpha^{1/2}\mathcal{O}+0.5(\bar{X}^a)^T(C+e^{-r}\bar{\gamma}\Sigma^X)^{-1/2})(C+e^{-r}\bar{\gamma}\Sigma^X)^{-1/2} \equiv \Theta\,.$$

Q.E.D.

Lemma 12 *Let* $\tilde{Q} = (\mathrm{Id} - S)QC^{1/2}$. *Then,*

$$\tilde{Q}\tilde{Q}^T = \tilde{Q}\tilde{X}\,,\ \mathrm{Im}\tilde{Q} \subset (\mathrm{Id} - S)\mathbb{R}^N\,, \tag{15}$$

where $\tilde{X} = C^{-1/2}\bar{X}^a(\mathrm{Id} - S)\,.$

Let J be an arbitrary orthogonal matrix and P be an orthogonal projection on a subspace

$\mathcal{X} \subset \mathbb{R}^N$ such that $PJ\tilde{X}$ is symmetric and positive semi-definite, and $PJ\tilde{X}\mathbb{R}^N \subset (\mathrm{Id}-S)\mathbb{R}^N$. Then, the general solution to (15) is given by $\tilde{Q} = PJ\tilde{X}J$.

Proof. Let $\tilde{Q} = HJ$ be the polar decomposition of $\tilde{Q}$, with $H = (\tilde{Q}\tilde{Q}^T)^{1/2}$ being the unique positive semi-definite square root of $\tilde{Q}\tilde{Q}^T$, and where J is an orthogonal matrix. Then, we get $H^2 = H\hat{X}$ with $\hat{X} = J\tilde{X}$. Since H is symmetric positive semi-definite, we get that $\hat{X}h = 0$ always implies $Hh = 0$ which implies that $H = P\hat{X}$ for some matrix P. Furthermore , if $\bar{X}^a$ is non-degenerate, then $\ker \hat{X}$ coincides with the image of S, which in turn coincides with the span of the vector $\mathbf{1}$. Substituting, we get the identity $P\hat{X}P\hat{X} = P\hat{X}^2$. Furthermore, without loss of generality, we may assume that $Ph = 0$ for all h with $\hat{X}h = 0$. Let us decompose $\mathbb{R}^N$ into the orthogonal sum of $\ker \hat{X}$ and its orthogonal complement. In this decomposition, the matrices take the form

$$P = \begin{pmatrix} P_1 & 0 \\ P_2 & 0 \end{pmatrix}, \ \hat{X} = \begin{pmatrix} X_1 & 0 \\ X_2 & 0 \end{pmatrix}$$

where X_1 is invertible. Since $P\hat{X}$ must be symmetric, we get $P_2 = 0$, and $P_1X_1P_1 = P_1X_1$. This implies that X_1 needs to map the kernel of P_1 into itself. Let us now perform the same block decomposition as above, but now for the matrices P_1 and X_1, with respect to the kernel of P_1. Then,

$$P_1 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix}, \ X_1 = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$$

and the symmetry of P_1X_1 implies that $P_{21} = 0$. Since both P_{11} and X_{11} are invertible, $P_1X_1P_1 = P_1X_1$ implies $P_{11} = \mathrm{Id}$ and hence P is the orthogonal projection onto its image. Furthermore, the symmetry of P_1X_1 is equivalent to the symmetry of X_{11}, which is in turn equivalent to the symmetry of $P\hat{X}$. Q.E.D.

Proof of Lemma 10. Equilibrium equation is

$$QCQ^T \;=\; Q\bar{X}^a \;-\; \bar{A}^F \;-\; Q\bar{\gamma}\Sigma^X Q^T S$$

Denote by A and B the left- and right-hand sides of these identities, respectively. The key observation is that, for any two matrices A, B the identity $A = B$ is equivalent to the system of four identities

$$SAS = SBS,\ (\mathrm{Id}-S)A(\mathrm{Id}-S) = (\mathrm{Id}-S)B(\mathrm{Id}-S),\ (\mathrm{Id}-S)AS = (\mathrm{Id}-S)BS,\ SA(\mathrm{Id}-S) = SB(\mathrm{Id}-S)\,.$$

Substituting $Q = R + \Theta$, we notice that equilibrium equations for R and Θ (equations (9) and (11)). precisely give the identities $SAS = SBS$, $(\mathrm{Id}-S)A(\mathrm{Id}-S) = (\mathrm{Id}-S)B(\mathrm{Id}-S)$, while (12) gives the last two conditions, $(\mathrm{Id}-S)AS = (\mathrm{Id}-S)BS$, $SA(\mathrm{Id}-S) = SB(\mathrm{Id}-S)$. Q.E.D.

Proof of Proposition 11. An equilibrium is symmetric if and only if $R = 0$. Equations (12) imply that this is only possible if $(\mathrm{Id} - S)\bar{A}^F = 0$ (symmetric financial frictions), and $\Theta\bar{X}^a(\mathrm{Id} - S) = 0$. Thus, by Theorem 7, existence of equilibria is equivalent to (13). If real costs are symmetric, then $(\bar{X}^a(\mathrm{Id} - S)) = 0$ and hence we have a continuum of equilibria. For generic parameter values, $\mathrm{span}(\bar{X}^a(\mathrm{Id} - S))$ has dimension $N - 1$, and hence there is at most a unique (up to a constant multiple). β in the set (13). Q.E.D.

Proof of Proposition 5. Equilibrium system takes the form

$$\begin{aligned}
b_{1,1}q_{1,1} &= \bar{X}^a_{1,1} + \frac{q_{1,2}}{\Delta}\bar{a}^F_2 - 0.5e^{-r}\bar{\gamma}(\Sigma_{1,1}(q_{1,1}+q_{2,1}) + \Sigma_{1,2}(q_{1,2}+q_{2,2}))\\
b_{2,1}q_{2,1} &= \bar{X}^a_{2,1} + \frac{q_{1,2}}{\Delta}\bar{a}^F_2 - 0.5e^{-r}\bar{\gamma}(\Sigma_{1,1}(q_{1,1}+q_{2,1}) + \Sigma_{1,2}(q_{1,2}+q_{2,2}))\\
b_{1,2}q_{1,2} &= \bar{X}^a_{1,2} - \frac{q_{1,1}}{\Delta}\bar{a}^F_2 - 0.5e^{-r}\bar{\gamma}(\Sigma_{1,2}(q_{1,1}+q_{2,1}) + \Sigma_{2,2}(q_{1,2}+q_{2,2}))\\
b_{2,2}q_{2,2} &= \bar{X}^a_{2,2} - \frac{q_{1,1}}{\Delta}\bar{a}^F_2 - 0.5e^{-r}\bar{\gamma}(\Sigma_{1,2}(q_{1,1}+q_{2,1}) + \Sigma_{2,2}(q_{1,2}+q_{2,2}))
\end{aligned}$$

where $\Delta = q_{1,1}q_{2,2} - q_{1,2}q_{2,1}$. Using Assumption 1 and the identities $q_{2,1} = b_{2,1}^{-1}(b_{1,1}q_{1,1} + \bar{X}^a_{2,1} - \bar{X}^a_{1,1}) \equiv q_{1,1} - y_1$, and $q_{2,2} = b_{2,2}^{-1}(b_{1,2}q_{1,2} + \bar{X}^a_{2,2} - \bar{X}^a_{1,2}) \equiv q_{1,2} + y_2$, we get the system

$$\begin{aligned} c_1 q_{1,1} &= \bar{X}^a_{1,1} + \frac{q_{1,2}}{q_{1,1}y_2 + q_{1,2}y_1}\bar{a}_2^F - 0.5e^{-r}\bar{\gamma}(\Sigma_{11}(2q_{1,1} - y_1) + \Sigma_{1,2}(2q_{1,2} + y_2)) \\ c_2 q_{1,2} &= \bar{X}^a_{1,2} - \frac{q_{1,1}}{q_{1,1}y_2 + q_{1,2}y_1}\bar{a}_2^F - 0.5e^{-r}\bar{\gamma}(\Sigma_{12}(2q_{1,1} - y_1) + \Sigma_{2,2}(2q_{1,2} + y_2)) \end{aligned}$$

That is,

$$\begin{pmatrix} q_{1,1} \\ q_{1,2} \end{pmatrix} = F \begin{pmatrix} \tilde{X}^a_{1,1} + \frac{q_{1,2}}{q_{1,1}y_2 + q_{1,2}y_1}\bar{a}_2^F \\ \tilde{X}^a_{1,2} - \frac{q_{1,1}}{q_{1,1}y_2 + q_{1,2}y_1}\bar{a}_2^F \end{pmatrix} \tag{16}$$

where

$$F = \begin{pmatrix} c_1 + e^{-r}\bar{\gamma}\Sigma_{11} & e^{-r}\bar{\gamma}\Sigma_{12} \\ e^{-r}\bar{\gamma}\Sigma_{12} & c_2 + e^{-r}\bar{\gamma}\Sigma_{22} \end{pmatrix}^{-1}, \quad \begin{pmatrix} \tilde{X}^a_{1,1} \\ \tilde{X}^a_{1,2} \end{pmatrix} = \begin{pmatrix} \bar{X}^a_{1,1} - 0.5e^{-r}\bar{\gamma}(\Sigma_{11}(-y_1) + \Sigma_{1,2}(y_2)) \\ \bar{X}^a_{1,2} - 0.5e^{-r}\bar{\gamma}(\Sigma_{12}(-y_1) + \Sigma_{2,2}(y_2)) \end{pmatrix}.$$

Dividing the first equation in (16) with the second one, we arrive at the following equation for the quotient $Z = q_{1,1}/q_{1,2}$:

$$Z = \frac{q^*_{1,1} + F_{1,1}\frac{1}{Zy_2 + y_1}\bar{a}_2^F - F_{1,2}\frac{Z}{Zy_2 + y_1}\bar{a}_2^F}{q^*_{1,2} + F_{1,2}\frac{1}{Zy_2 + y_1}\bar{a}_2^F - F_{2,2}\frac{Z}{Zy_2 + y_1}\bar{a}_2^F},$$

and we arrive at the required quadratic equation for Z :

$$Z^2(y_2 q^*_{1,2} - F_{2,2}\bar{a}_2^F) + Z(y_1 q^*_{1,2} - y_2 q^*_{1,1} + 2F_{1,2}\bar{a}_2^F) - y_1 q^*_{1,1} - F_{1,1}\bar{a}_2^F = 0.$$

Thus,

$$Z = \frac{-(y_1 q^*_{1,2} - y_2 q^*_{1,1} + 2F_{1,2}\bar{a}_2^F) \pm \left((y_1 q^*_{1,2} - y_2 q^*_{1,1} + 2F_{1,2}\bar{a}_2^F)^2 + 4(y_2 q^*_{1,2} - F_{2,2}\bar{a}_2^F)(y_1 q^*_{1,1} + F_{1,1}\bar{a}_2^F)\right)^{1/2}}{2(y_2 q^*_{1,2} - F_{2,2}\bar{a}_2^F)}$$

Here,

$$q^*_{1,1} \;=\; F_{1,1}\tilde{X}^a_{1,1} + F_{1,2}\tilde{X}^a_{1,2},\;\; q^*_{1,2} \;=\; F_{1,2}\tilde{X}^a_{1,1} + F_{2,2}\tilde{X}^a_{1,2}\,.$$

Q.E.D.

Proof of Corollary 6. Note first that the positivity of $q_{i,j}$ always follows automatically from $Z > 0$ because, by assumption, $q^*_{i,j} > 0$. Indeed, we have $q_{1,1} > q^*_{1,1}$ is then always positive, and hence $q_{1,2} = q_{1,1} - y_1 > q^*_{1,2} > 0$. But then $q_{1,2} = q_{1,1}/Z > 0$ and $q_{2,2} > q_{1,2} > 0$.

If $y_2 q^*_{1,2} > F_{2,2}\bar{a}^F_2$ then $Z_+ > 0$ by direct calculation, and the fragility result follows by direct calculation. The other case is similar.

Q.E.D.